

Lorentz Transformations

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Abstract. This paper describes a particularly didactic and transparent derivation of basic properties of the Lorentz group. The generators for rotations and boosts along an arbitrary direction, as well as their commutation relations, are written as functions of the unit vectors that define the axis of rotation or the direction of the boost (an approach that can be compared with the one that in electrodynamics, works with the electric and magnetic fields instead of the Maxwell stress tensor). For finite values of the angle of rotation or the boost's velocity (collectively denoted by \mathcal{V}), the existence of an exponential expansion for the coordinate transformation's matrix, \mathcal{M} , in terms of $G\mathcal{V}$ with G being the generator, requires that the matrix's derivative with respect to \mathcal{V} be equal to $G\mathcal{M}$. This condition can only be satisfied if the transformation is additive as it is indeed the case for rotations, but not for velocities. If it is assumed, however, that for boosts such an expansion exists, with $\mathcal{V} = \mathcal{V}(v)$, v being the boost's velocity, and if the above condition is imposed on the boost's matrix, then its expression in terms of hyperbolic $\cosh(\mathcal{V})$ and $\sinh(\mathcal{V})$ is recovered with $\mathcal{V} (= \tanh^{-1}(v))$.

A general Lorentz transformation can be written as an exponential containing the sum of a rotation and a boost, which to first order is equal to the product of a boost with a rotation. The calculations of the second and third order terms show that the equations for the generators used in this paper, allow to reliably infer the expressions for the higher order generators, without having recourse to the commutation relations.

The transformation matrices for Weyl spinors are derived for finite values of the rotation and velocity, and field representations, leading to the expression for the angular momentum operator, are studied, as well as the Lorentz transformation properties of the generators.

1. Rotations and Boosts. Arbitrary Axes. By definition, a Lorentz transformations of coordinates, namely,

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} = (\delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}) x^{\nu} \quad (1)$$

leaves the metric,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = dt^2 - dx^2 - dy^2 - dz^2, \quad (2)$$

invariant. It is straightforward to show that this condition imposes the following constraints on the coefficients Λ^{μ}_{ν} ,

$$g_{\mu\nu} = g_{\alpha\beta} \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu}, \quad \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \quad (3)$$

where the second equation is the linear version of the first one for small values of $\omega_{\mu\nu} = g_{\mu\alpha}(\Lambda^\alpha_\nu - \delta^\alpha_\nu)$. The axis of a rotation by an angle Θ , counter clockwise is positive, will be denoted by $\mathbf{r} = (r_x, r_y, r_z) = (r^1, r^2, r^3)$, a unit vector. The matrix associated with this Lorentz transformation is ($\Omega = 1 - \cos \Theta$),

$$\Lambda_{rot}(\Theta, \mathbf{r}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_x^2 + (1 - r_x^2) \cos \Theta & r_x r_y \Omega - r_z \sin \Theta & r_x r_z \Omega + r_y \sin \Theta \\ 0 & r_x r_y \Omega + r_z \sin \Theta & r_y^2 + (1 - r_y^2) \cos \Theta & r_y r_z \Omega - r_x \sin \Theta \\ 0 & r_x r_z \Omega - r_y \sin \Theta & r_y r_z \Omega + r_x \sin \Theta & r_z^2 + (1 - r_z^2) \cos \Theta \end{pmatrix} \quad (4)$$

The matrix associated with a boost with a velocity v and a direction defined by the unit vector $\mathbf{b} = (b_x, b_y, b_z) = (b^1, b^2, b^3)$ is given by, ($\gamma = (1 - v^2)^{-1/2}$, c has been set equal to one),

$$\Lambda_{boost}(v, \mathbf{b}) = \begin{pmatrix} \gamma & v b_x \gamma & v b_y \gamma & v b_z \gamma \\ v b_x \gamma & 1 + (\gamma - 1) b_x^2 & (\gamma - 1) b_x b_y & (\gamma - 1) b_x b_z \\ v b_y \gamma & (\gamma - 1) b_y b_x & 1 + (\gamma - 1) b_y^2 & (\gamma - 1) b_y b_z \\ v b_z \gamma & (\gamma - 1) b_z b_x & (\gamma - 1) b_z b_y & 1 + (\gamma - 1) b_z^2 \end{pmatrix} \quad (5)$$

2. Generators and Commutation Relations. It follows from Eqs.(4) and (5), that the matrix for a Lorentz transformation, combining a boost and a rotation, and linear in v and Θ , is given by,

$$\Lambda = 1 + \begin{pmatrix} 0 & v b^1 & v b^2 & v b^3 \\ v b^1 & 0 & -\Theta r^3 & \Theta r^2 \\ v b^2 & \Theta r^3 & 0 & -\Theta r^1 \\ v b^3 & -\Theta r^2 & \Theta r^1 & 0 \end{pmatrix} = 1 + \Theta G_{rot}(\mathbf{r}) + v G_{boost}(\mathbf{b}) \quad (6)$$

where,

$$G_{rot}(\mathbf{r}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r^3 & r^2 \\ 0 & r^3 & 0 & -r^1 \\ 0 & -r^2 & r^1 & 0 \end{pmatrix}, \quad (7)$$

and

$$G_{boost}(\mathbf{b}) = \begin{pmatrix} 0 & b^1 & b^2 & b^3 \\ b^1 & 0 & 0 & 0 \\ b^2 & 0 & 0 & 0 \\ b^3 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

are the generators for a rotation around \mathbf{r} , and for a boost along \mathbf{b} , respectively. In the right-hand side of Eq.(6), Θ and v should be interpreted as diagonal matrices. From Eqs.(7) and (8) one finds that the commutation rules for the rotation-rotation generators are,

$$[G_{rot}(\mathbf{r}), G_{rot}(\mathbf{r}')] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 r'^1 - r^1 r'^2 & r^3 r'^1 - r^1 r'^3 \\ 0 & r^1 r'^2 - r^2 r'^1 & 0 & r^3 r'^2 - r^2 r'^3 \\ 0 & r^1 r'^3 - r^3 r'^1 & r^2 r'^3 - r^3 r'^2 & 0 \end{pmatrix} \quad (9)$$

which obviously defines a rotation. The coefficients of this matrix are the components of the vector $\mathbf{r} \wedge \mathbf{r}'$ and clearly,

$$[G_{rot}(\mathbf{r}), G_{rot}(\mathbf{r}')] = G_{rot}(\mathbf{r} \wedge \mathbf{r}') \quad (10)$$

For the boost-boost generators one obtains,

$$[G_{boost}(\mathbf{b}), G_{boost}(\mathbf{b}')] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b^1 b'^2 - b^2 b'^1 & b^1 b'^3 - b^3 b'^1 \\ 0 & b^2 b'^1 - b^1 b'^2 & 0 & b^2 b'^3 - b^3 b'^2 \\ 0 & b^3 b'^1 - b^1 b'^3 & b^3 b'^2 - b^2 b'^3 & 0 \end{pmatrix} \quad (11)$$

also a rotation, and

$$[G_{boost}(\mathbf{b}), G_{boost}(\mathbf{b}')] = -G_{rot}(\mathbf{b} \wedge \mathbf{b}') \quad (12)$$

For the rotation-boost, the commutation relations are,

$$[G_{rot}(\mathbf{r}), G_{boost}(\mathbf{b})] = \begin{pmatrix} 0 & r^2 b^3 - r^3 b^2 & r^3 b^1 - r^1 b^3 & r^1 b^2 - r^2 b^1 \\ r^2 b^3 - r^3 b^2 & 0 & 0 & 0 \\ r^3 b^1 - r^1 b^3 & 0 & 0 & 0 \\ r^1 b^2 - r^2 b^1 & 0 & 0 & 0 \end{pmatrix} \quad (13)$$

$$[G_{rot}(\mathbf{r}), G_{boost}(\mathbf{b})] = G_{boost}(\mathbf{r} \wedge \mathbf{b}) \quad (14)$$

As expected the commutation relations in Eqs. (10), (12), and (14) are in agreement with those satisfied by the J^i and K^i generators derived from a Lorentz transformation as follows,

$$\begin{aligned} x'^\alpha &= (\delta_\beta^\alpha + \omega_\beta^\alpha) x^\beta = (\delta_\beta^\alpha + g^{\alpha\mu} \omega_{\mu\beta}) x^\beta = (\delta_\beta^\alpha + g^{\alpha\mu} \delta_\beta^\nu \omega_{\mu\nu}) x^\beta \\ &= \left(\delta_\beta^\alpha + \frac{1}{2} \omega_{\mu\nu} (g^{\alpha\mu} \delta_\beta^\nu - g^{\alpha\nu} \delta_\beta^\mu) \right) x^\beta = \left(\delta_\beta^\alpha + \frac{1}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha_\beta \right) x^\beta \end{aligned} \quad (15)$$

In Eq.(15), the fifth term appeals to the linear Lorentz condition, $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$, followed by a relabeling ($\mu \rightleftharpoons \nu$), and the sixth term shows that the expression for the generators is given by,

$$(J^{\mu\nu})^\alpha_\beta = g^{\alpha\mu} \delta_\beta^\nu - g^{\alpha\nu} \delta_\beta^\mu \quad (16)$$

Define,

$$J^i = \frac{1}{2} \epsilon^{ijk} J^{jk}, \quad K^i = J^{io}, \quad \theta^i = \frac{1}{2} \epsilon^{ijk} \omega^{jk}, \quad \eta^i = \omega^{io} \quad (17)$$

Then, J^i , K^i satisfy the same commutation rules (namely Eqs.(10), (12) and (14)), than $G_{rot}(\mathbf{r})$ and $G_{boost}(\mathbf{b})$ with \mathbf{r} and \mathbf{b} being equal to \mathbf{e}^1 , \mathbf{e}^2 , \mathbf{e}^3 , the unit basis vectors. It is now straightforward to show that,

$$x'^\alpha = \left(\delta_\beta^\alpha + \frac{1}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha_\beta \right) x^\beta = \left(\delta_\beta^\alpha - (\mathbf{J} \cdot \boldsymbol{\theta} - \mathbf{K} \cdot \boldsymbol{\eta})^\alpha_\beta \right) x^\beta \quad (18)$$

In this equation the first minus sign would be a plus one, if the metric coefficients, defined in Eq.(2), are reversed. The matrix for the Lorentz transformation defined in Eq.(18) is found to be,

$$\Lambda = 1 + \begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3 \\ \eta^1 & 0 & -\theta^3 & \theta^2 \\ \eta^2 & \theta^3 & 0 & -\theta^1 \\ \eta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} \quad (19)$$

A comparison with Eq.(6) reveals that $vb^i = \eta^i$ and $\Theta r^i = \theta^i$. Because \mathbf{b} and \mathbf{r} are unit vectors it follows that,

$$v = \pm ((\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2)^{1/2}, \quad \Theta = \pm ((\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2)^{1/2} \quad (20)$$

and in consequence,

$$r^i = \pm \theta^i / |\boldsymbol{\theta}| \quad b^i = \pm \eta^i / |\boldsymbol{\eta}| \quad (21)$$

Therefore θ^i, η^i determine the axis of rotation and boost, but not the sense of rotation nor the direction of the boost

3. Finite rotations and boosts. The matrix for a general Lorentz transformation is usually written as,

$$\boldsymbol{\Lambda} = \exp\{-\mathbf{J} \cdot \boldsymbol{\theta} + \mathbf{K} \cdot \boldsymbol{\eta}\} \quad (22)$$

Working with the generators defined in Eqs.(7) and (8), this expression for $\boldsymbol{\Lambda}$ is replaced by,

$$\boldsymbol{\Lambda} = \exp\{\Theta G_{rot}(\mathbf{r}) + v G_{boost}(\mathbf{b})\}, \quad (23)$$

Eq.(23) is reminiscent of a Taylor expansion for a function of two variables, namely,

$$f(\Theta, v) = \exp\{\Theta \partial_\Theta + v \partial_v\} f(\Theta, v) \quad (24)$$

where the derivatives play the role of the generators. The derivatives, unlike generators, do however commute.

a) *Rotations.* It needs to be shown that $\boldsymbol{\Lambda}_{rot}(\Theta, \mathbf{r})$, as given by Eq.(4), is equal to,

$$\exp\{\Theta G_{rot}(\mathbf{r})\} = \boldsymbol{\Lambda} = 1 + \Theta G_{rot}(\mathbf{r}) + \frac{1}{2} \Theta^2 G_{rot}(\mathbf{r})^2 + \frac{1}{3!} \Theta^3 G_{rot}(\mathbf{r})^3 + \dots \quad (25)$$

The relabeling $\boldsymbol{\Lambda}_{rot}(\Theta, \mathbf{r}) = \boldsymbol{\Lambda}$ is for notational convenience. For the above expansion to be valid it is necessary that,

$$\partial_\Theta \boldsymbol{\Lambda} = G_{rot}(\mathbf{r}) \boldsymbol{\Lambda} \quad (26)$$

From Eq.(4) it is found that,

$$\partial_\Theta \boldsymbol{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(1 - r_x^2) \sin \Theta & r_x r_y \sin \Theta - r_z \cos \Theta & r_x r_z \sin \Theta + r_y \cos \Theta \\ 0 & r_x r_y \sin \Theta + r_z \cos \Theta & -(1 - r_y^2) \sin \Theta & r_y r_z \sin \Theta - r_x \cos \Theta \\ 0 & r_x r_z \sin \Theta - r_y \cos \Theta & r_y r_z \sin \Theta + r_x \cos \Theta & -(1 - r_z^2) \sin \Theta \end{pmatrix} \quad (27)$$

and it is straightforward to verify that Eq.(26) is indeed satisfied. Therefore Eq.(25) is the correct expression for finite rotations.

The expression for $\boldsymbol{\Lambda}_{rot}(\Theta, \mathbf{r})$ in Eq.(4) can be derived from *just the first three terms* of the expansion for $\boldsymbol{\Lambda}_{rot}(\Theta, \mathbf{r})$ in Eq.(25): the term in Θ is replaced by $\sin \Theta$ and the next one, in $\frac{1}{2} \Theta^2$, by Ω (notice that $\Omega = 1 - \cos \Theta = \frac{1}{2} \Theta^2 + \mathcal{O}(\Theta^4)$). How is this possible?, an alert reader is bound to ask, Θ does not stand in isolation, instead it is multiplied by $G_{rot}(\mathbf{r})$, which surely cannot be ignored. Very good question indeed, that allows however for an apparent answer:

$$\begin{aligned}
G_{rot}(\mathbf{r}) &= -G_{rot}(\mathbf{r})^3 = G_{rot}(\mathbf{r})^5 = -G_{rot}(\mathbf{r})^7 = \dots, & G_{boost}(\mathbf{b}) &= G_{boost}(\mathbf{b})^3 = \\
G_{boost}(\mathbf{b})^5 &= G_{boost}(\mathbf{b})^7 = \dots, & G_{rot}(\mathbf{r})^2 &= -G_{rot}(\mathbf{r})^4 = G_{rot}(\mathbf{r})^6 = -G_{rot}(\mathbf{r})^8 = \dots \\
G_{boost}(\mathbf{b})^2 &= G_{boost}(\mathbf{b})^4 = G_{boost}(\mathbf{b})^6 = G_{boost}(\mathbf{b})^8 = \dots
\end{aligned} \tag{28}$$

It is important to keep in mind that Eq.(26) implies the differential version of the addition of rotations, namely, $\mathbf{\Lambda}_{rot}(\Theta + d\Theta) = \mathbf{\Lambda}_{rot}(\Theta)\mathbf{\Lambda}_{rot}(d\Theta)$. Indeed,

$$\begin{aligned}
\mathbf{\Lambda}_{rot}(\Theta + d\Theta) &= \mathbf{\Lambda}_{rot}(\Theta) + d\Theta \partial_{\Theta} \mathbf{\Lambda}_{rot}(\Theta) = \mathbf{\Lambda}_{rot}(\Theta) + d\Theta G_{rot}(\mathbf{r}) \mathbf{\Lambda}_{rot}(\Theta) = \\
&= (1 + d\Theta G_{rot}(\mathbf{r})) \mathbf{\Lambda}_{rot}(\Theta) = \mathbf{\Lambda}_{rot}(d\Theta) \mathbf{\Lambda}_{rot}(\Theta)
\end{aligned} \tag{29}$$

b) *Boosts*. No expansion for $\mathbf{\Lambda}_{boost}$ analogous to the one for $\mathbf{\Lambda}_{rot}$, i.e. with $\Theta \rightarrow v$ and $G_{rot}(\mathbf{r}) \rightarrow G_{boost}(\mathbf{b})$, can exist, because as shown above, it would imply that $\mathbf{\Lambda}_{boost}(v + dv) = \mathbf{\Lambda}_{boost}(v)\mathbf{\Lambda}_{boost}(dv)$, in contradiction with the non additive property of velocities in Relativity. Assume therefore instead that an expansion of the form (in the text of this paper, and contrary to the Abstract, the symbol \mathcal{V} will designate a function of the velocity, v).

$$\mathcal{V} = \mathcal{V}(v), \quad \mathbf{\Lambda}_{boost}(\mathcal{V}) = \exp\{\mathcal{V}G_{boost}(\mathbf{b})\} = 1 + \mathcal{V}G_{boost}(\mathbf{b}) + \frac{1}{2}\mathcal{V}^2 G_{boost}(\mathbf{b})^2 + \dots \tag{30}$$

Then,

$$\partial_{\mathcal{V}} \mathbf{\Lambda}_{boost}(\mathcal{V}) = \mathbf{\Lambda}_{boost}(\mathcal{V}) G_{boost}(\mathbf{b}) \tag{31}$$

is satisfied. Imposing this condition on $\mathbf{\Lambda}_{boost} = \mathbf{\Lambda}_{boost}(\mathbf{b}, v)$, as given by Eq.(5), leads to the following relations,

$$\frac{d\gamma}{d\mathcal{V}} = v\gamma, \quad \frac{d(v\gamma)}{d\mathcal{V}} = \gamma = (1 - v^2)^{-1/2}, \tag{32}$$

It is now straightforward to derive an equation for $\partial_v \mathcal{V}(v)$,

$$\begin{aligned}
\frac{d\gamma}{d\mathcal{V}} &= \frac{d(1 - v^2)^{-1/2}}{d\mathcal{V}} = \gamma^2 v \frac{dv}{d\mathcal{V}} = \gamma^2 \frac{d\gamma}{d\mathcal{V}} \frac{dv}{d\mathcal{V}} \rightarrow 1 = \gamma^2 \frac{dv}{d\mathcal{V}} \rightarrow \\
\frac{d\mathcal{V}}{dv} &= \frac{1}{1 - v^2} = \frac{d}{dv} \tanh^{-1} v
\end{aligned} \tag{33}$$

Therefore,

$$\gamma = \cosh(\mathcal{V}), \quad \mathcal{V}(v) = \tanh^{-1}(v) = v + v^3/3 + v^5/5 + \dots \tag{34}$$

The matrix for the boost, namely $\mathbf{\Lambda}_{boost}(v, \mathbf{b})$ in Eq.(5), can now be written,

$$\mathbf{\Lambda}_{boost}(\mathcal{V}, \mathbf{b}) = \begin{pmatrix} \gamma & b_x \sinh(\mathcal{V}) & b_y \sinh(\mathcal{V}) & b_z \sinh(\mathcal{V}) \\ b_x \sinh(\mathcal{V}) & 1 + (\gamma - 1)b_x^2 & (\gamma - 1)b_x b_y & (\gamma - 1)b_x b_z \\ b_y \sinh(\mathcal{V}) & (\gamma - 1)b_y b_x & 1 + (\gamma - 1)b_y^2 & (\gamma - 1)b_y b_z \\ b_z \sinh(\mathcal{V}) & (\gamma - 1)b_z b_x & (\gamma - 1)b_z b_y & 1 + (\gamma - 1)b_z^2 \end{pmatrix} \tag{35}$$

It is again possible to derive the expression for $\mathbf{\Lambda}_{boost}(v, \mathbf{b})$, (cf. Eq.(5), and Eq.(28)) from the first three terms of Eq.(30). In the linear term of this expansion, replace \mathcal{V} by $v\gamma$ (notice that $v\gamma = d\gamma/d\mathcal{V} = d \cosh(\mathcal{V})/d\mathcal{V} = \sinh \mathcal{V} = \mathcal{V} + \mathcal{O}(mcV^3)$), and

in the quadratic term, replace $\mathcal{V}^2/2$, by $\gamma - 1$ ($\gamma - 1 = \mathcal{V}^2/2 + \mathcal{O}(\mathcal{V}^4)$). The linear replacement reproduces the first line and the first column of Eq(5), whereas the quadratic replacement introduces the factor $\gamma - 1$ present in $\mathbf{\Lambda}_{boost}(v, \mathbf{b})$.

c) *Rotations and Boosts*. Given the Lorentz transformation, $\mathcal{LT} = \exp\{\Theta G_{rot}(\mathbf{r}) + \mathcal{V} G_{boost}(\mathbf{b})\}$, the problem at hand is to find the product of rotations and boosts equivalent to \mathcal{LT} . To first order in Θ and \mathcal{V} , the transformation \mathcal{LT} and $\exp\{\Theta G_{rot}(\mathbf{r})\} \times \exp\{\mathcal{V} G_{boost}(\mathbf{b})\}$ are of course equal, but to second order, the terms in $\Theta\mathcal{V}$ differ. Therefore a relation of the following form must exist,

$$\exp\{\Theta G_{rot}(\mathbf{r}) + \mathcal{V} G_{boost}(\mathbf{b})\} = \exp\{\Theta G_{rot}(\mathbf{r})\} \times \exp\{\mathcal{V} G_{boost}(\mathbf{b})\} \times \exp\{\alpha \Theta \mathcal{V} G_x(\mathbf{v})\} \quad (36)$$

where α is a constant to be determined, x stands for a boost or rotation, and \mathbf{v} for the unit vector that defines the axis of rotation or the boost's direction. The expression for $G_x(\mathbf{v})$ can be easily inferred: \mathbf{v} must be a combination of the vectors \mathbf{r} and \mathbf{b} and the obvious choice is $\mathbf{v} = \mathbf{r} \wedge \mathbf{b}$, which defines a boost, as Eq.(14) shows. Therefore $G_x(\mathbf{v}) = G_{boost}(\mathbf{r} \wedge \mathbf{b})$. The equality of the terms in $\Theta\mathcal{V}$ in Eq.(36) leads then to,

$$\frac{1}{2} (G_{rot} G_{boost} + G_{boost} G_{rot}) = G_{rot} G_{boost} + \alpha G_{boost}(\mathbf{r} \wedge \mathbf{b}) \quad (37)$$

which can indeed be satisfied with $\alpha = -1/2$. The notation (to be used from now on) has been simplified: $G_{rot} = G_{rot}(\mathbf{r})$ and $G_{boost} = G_{boost}(\mathbf{b})$. Eq.(37) can also be written,

$$-\frac{1}{2} [G_{rot}, G_{boost}] = \alpha G_x(\mathbf{v}) \quad (38)$$

If no prior knowledge of $G_x(\mathbf{v})$ had been assumed, this equation would have determined α and $G_x(\mathbf{v})$, which, because these are the commutation relations, *cannot be other than* $G_{boost}(\mathbf{r} \wedge \mathbf{b})$.

It is of interest to calculate the third order terms. There are now two terms, namely those in $\Theta^2\mathcal{V}$ and $\mathcal{V}^2\Theta$, that require to be balanced by two new Lorentz transformations which are determined by:

$$\begin{aligned} \exp\{\Theta G_{rot} + \mathcal{V} G_{boost}\} &= \exp\{\Theta G_{rot}\} \times \exp\{\mathcal{V} G_{boost}\} \times \exp\{-\frac{1}{2} \Theta \mathcal{V} G_{boost}(\mathbf{r} \wedge \mathbf{b})\} \times \\ &\times \exp\{\alpha \Theta^2 \mathcal{V} G_x(\mathbf{v})\} \times \exp\{\beta \Theta \mathcal{V}^2 G_y(\mathbf{v}')\} \end{aligned} \quad (39)$$

where, x, y stand for boost or rotation and \mathbf{v}, \mathbf{v}' for the vectors defining the axes of the transformations. It is again possible to infer the expressions for $G_x(\mathbf{v})$ and $G_y(\mathbf{v}')$. The obvious choices for \mathbf{v} and \mathbf{v}' are now $\mathbf{v} = \mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{b})$ and $\mathbf{v}' = \mathbf{b} \wedge (\mathbf{r} \wedge \mathbf{b})$. It is clear that \mathbf{v} with two \mathbf{r} s and one \mathbf{b} , should be associated with $\Theta^2\mathcal{V}$ and that \mathbf{v}' with two \mathbf{b} s and one \mathbf{r} , with $\mathcal{V}^2\Theta$. Keeping in mind that $\mathbf{r} \wedge \mathbf{b}$ is a boost, Eq.(14) suggests that x is a boost, and Eq.(12), that y is a rotation. Therefore,

$$G_x(\mathbf{v}) = G_{boost}(\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{b})) \quad G_y(\mathbf{v}') = G_{rot}(\mathbf{b} \wedge (\mathbf{r} \wedge \mathbf{b})) \quad (40)$$

The equality of the terms in $\Theta^2\mathcal{V}$ in Eq.(39) leads then to,

$$\begin{aligned} \frac{1}{6} (G_{rot}^2 G_{boost} + G_{rot} G_{boost} G_{rot} + G_{boost} G_{rot}^2) &= \frac{1}{2} (G_{rot}^2 G_{boost} - G_{rot} G_{boost}(\mathbf{r} \wedge \mathbf{b})) \\ &+ \alpha G_{boost}(\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{b})) \end{aligned} \quad (41)$$

If $\mathbf{r} = \mathbf{e}_z$ and $\mathbf{b} = \mathbf{e}_x$, then $\mathbf{r} \wedge \mathbf{b} = \mathbf{e}_y$ and $\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{b}) = -\mathbf{e}_x$. All the *three* dimensional matrices in Eq.(41) are then easily evaluated, and the value found for α is $1/6$. Similar calculations for the terms in $\Theta\mathcal{V}^2$ show that $\beta = 1/3$. If no prior knowledge for $G_x(\mathbf{v})$ is assumed, $G_x(\mathbf{v})$ is of course *determined* by the commutation relations from Eq.(41). Contrast this statement with the one appropriate to the second order terms, where $G_x(\mathbf{v})$ is a *member* of the commutation relations.

It is clear that the Lorentz transformation $\exp\{\Theta G_{rot}(\mathbf{r}) + \mathcal{V}G_{boost}(\mathbf{b})\}$, is a complex product of rotations and boosts, not shedding light (as $\exp\{\Theta G_{rot}(\mathbf{r}) + \mathcal{V}G_{boost}(\mathbf{b})\}$ do), on finite transformations.

There can be little doubt that the theory of group representations could be developed as well from the product of a boost and a rotation, namely $\exp\{\Theta G_{rot}(\mathbf{r})\} \times \exp\{\mathcal{V}G_{boost}(\mathbf{b})\}$.

4. Spin. Define,

$$J^\pm(\mathbf{u}) = i(G_{rot}(\mathbf{u}) \pm iG_{boost}(\mathbf{u}))/2 = (iG_{rot}(\mathbf{u}) \mp G_{boost}(\mathbf{u}))/2 \quad (42)$$

The commutation rules are given by,

$$\begin{aligned} [J^+(\mathbf{u}), J^+(\mathbf{u}')] &= iJ^+(\mathbf{u} \wedge \mathbf{u}') \\ [J^-(\mathbf{u}), J^-(\mathbf{u}')] &= iJ^-(\mathbf{u} \wedge \mathbf{u}') \\ [J^+(\mathbf{u}), J^-(\mathbf{u}')] &= 0 \end{aligned} \quad (43)$$

With the help of Eqs. (7) and (8), the calculation of $(J^\pm(\mathbf{u}))^2$ is straightforward. It is found that (\mathbf{I} is the unit matrix),

$$(J^\pm(\mathbf{u}))^2 = \mathbf{I}/4, \quad (44)$$

The two degrees of freedom associated with particles of spin $1/2$ suggests working with a representation for the generators on a vector space of dimension two. The Pauli matrices are given by,

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ M(\mathbf{u}) &= \boldsymbol{\sigma} \cdot \mathbf{u} = \begin{pmatrix} u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix} & M^2(\mathbf{u}) &= \mathbf{I} \end{aligned} \quad (45)$$

The commutation relations for the matrix $M(\mathbf{u})$, defined above, are given by,

$$[M(\mathbf{u}), M(\mathbf{u}')] = 2i \begin{pmatrix} U_z & U_x - iU_y \\ U_x + iU_y & -U_z \end{pmatrix} \quad \mathbf{U} = \mathbf{u} \wedge \mathbf{u}' \quad (46)$$

Define

$$\mathcal{G}_{rot}^L(\mathbf{u}) = \frac{1}{2i}M(\mathbf{u}) \quad \mathcal{G}_{boost}^L(\mathbf{u}) = \frac{1}{2}M(\mathbf{u}) \quad \mathcal{J}^+(\mathbf{u}) = 0 \quad (47)$$

$$\mathcal{G}_{rot}^R(\mathbf{u}) = \frac{1}{2i}M(\mathbf{u}) \quad \mathcal{G}_{boost}^R(\mathbf{u}) = -\frac{1}{2}M(\mathbf{u}) \quad \mathcal{J}^-(\mathbf{u}) = 0 \quad (48)$$

where the \mathcal{J}^\pm are defined as in Eq.(42) with the \mathcal{G} generators replacing the G ones. With the help of Eq.(46) it is straightforward to show that $\mathcal{G}_{rot}^L(\mathbf{u}), \mathcal{G}_{boost}^L(\mathbf{u})$ (and of course $\mathcal{G}_{rot}^R(\mathbf{u}), \mathcal{G}_{boost}^R(\mathbf{u})$ as well), obey the same commutation relations (namely

Eqs. (10), (12), and (14)), than $G_{rot}(\mathbf{u})$ and $G_{boost}(\mathbf{u})$. Eqs. (47) and (48) are two representations of the Lorentz group, known as the left- and right handed Weyl spinors, that can be labelled by $((\mathcal{J}^-)^2 = \frac{1}{2}, \mathcal{J}^+ = 0)$ and $(\mathcal{J}^- = 0, (\mathcal{J}^+)^2 = \frac{1}{2})$ respectively. It is of interest to calculate the series

$$\begin{aligned}\Lambda^L(\Theta, \mathbf{u}) &= 1 + \Theta \mathcal{G}_{rot}^L(\mathbf{u}) + \frac{1}{2} \Theta^2 \mathcal{G}_{rot}^L(\mathbf{u})^2 + \dots \\ \Lambda^L(\mathcal{V}, \mathbf{u}) &= 1 + \mathcal{V} \mathcal{G}_{boost}^L(\mathbf{u}) + \frac{1}{2} \mathcal{V}^2 \mathcal{G}_{boost}^L(\mathbf{u})^2 + \dots\end{aligned}\quad (49)$$

Because the square of the matrix in the expressions for $\mathcal{G}_{rot}^L(\mathbf{u})$ and $\mathcal{G}_{boost}^L(\mathbf{u})$ is equal to the unit matrix, the calculations are straightforward. It is found,

$$\Lambda^L(\Theta, \mathbf{u}) = \begin{pmatrix} \cos(\Theta/2) - iu_z \sin(\Theta/2) & -i \sin(\Theta/2)(u_x - iu_y) \\ -i \sin(\Theta/2)(u_x + iu_y) & \cos(\Theta/2) + iu_z \sin(\Theta/2) \end{pmatrix} \quad (50)$$

$$\Lambda^L(\mathcal{V}, \mathbf{u}) = \begin{pmatrix} \cosh(\mathcal{V}/2) + u_z \sinh(\mathcal{V}/2) & \sinh(\mathcal{V}/2)(u_z - iu_y) \\ \sinh(\mathcal{V}/2)(u_z + iu_y) & \cosh(\mathcal{V}/2) - u_z \sinh(\mathcal{V}/2) \end{pmatrix} \quad (51)$$

For finite rotations and boosts, these matrices define the transformations for the left handed Weyl spinor components, as $\Lambda_{rot}(\Theta, \mathbf{r})$ and $\Lambda_{boost}(\mathcal{V}, \mathbf{b})$ (in Eqs. (4) and (35)), do for the coordinates of a Lorentz transformation. For the right handed Weyl spinor, $\Lambda^R(\Theta, \mathbf{u})$ remains unchanged, whereas \mathcal{V} is replaced by $-\mathcal{V}$ in the expression for $\Lambda^R(\mathcal{V}, \mathbf{u})$.

4. Field Representations. Let $T^{\mu\nu}$ (μ is line, and ν is column) be the antisymmetric tensor associated with the vectors \mathbf{r} and \mathbf{b} ($T^{0i} = b^i, T^{i0} = -b^i$ and the rotational components as in Eq.(7)), then

$$T^\mu{}_\nu = - \begin{pmatrix} 0 & b^1 & b^2 & b^3 \\ b^1 & 0 & -r^3 & r^2 \\ b^2 & r^3 & 0 & -r^1 \\ b^3 & -r^2 & r^1 & 0 \end{pmatrix} \quad (52)$$

Indeed, because of Eq.(2), $T^0{}_i = g_{i\alpha} T^{0\alpha} = g_{ii} T^{0i} = -T^{0i} = -b^i$. On the other hand, $T^i{}_0 = g_{0\alpha} T^{i\alpha} = T^{i0} = -b^i$, i.e., the boost components of $T^{\mu\nu}$ become symmetric, whereas the rotation components remain antisymmetric but change sign. The matrix in Eq.(52) is equal to the sum of $G_{rot}(\mathbf{r})$ and $G_{boost}(\mathbf{b})$ as given by Eqs. (7) and (8).

We define therefore the following operators,

$$A_{rot}(\mathbf{r}, x) = -G_{rot}(\mathbf{r})^\mu{}_\nu x^\nu \partial_\mu \quad A_{boost}(\mathbf{b}, x) = -G_{boost}(\mathbf{b})^\mu{}_\nu x^\nu \partial_\mu \quad (53)$$

The proof that $A_{rot}(\mathbf{r}, x)$ and $A_{boost}(\mathbf{b}, x)$ satisfy the same commutation rules than $G_{rot}(\mathbf{r})$ and $G_{boost}(\mathbf{b})$, namely Eqs.(10), (12), (14), is straightforward; it appeals to the relations, $G_{rot}(\mathbf{r})^\mu{}_\nu = -G_{rot}(\mathbf{r})^\nu{}_\mu$, and $G_{boost}(\mathbf{b})^\mu{}_\nu = G_{boost}(\mathbf{b})^\nu{}_\mu$, and involves some relabeling. It should be stressed that the minus signs in Eqs. (53), essential for this proof, can be firmly justified. The explicit expressions for $A_{rot}(\mathbf{r}, x)$ and $A_{boost}(\mathbf{b}, x)$ are $(t, x, y, z = x^\mu)$,

$$\begin{aligned}A_{rot}(\mathbf{r}, x) &= -(r_x(y\partial_z - z\partial_y) + r_y(z\partial_x - x\partial_z) + r_z(x\partial_y - y\partial_x)) \\ A_{boost}(\mathbf{b}, x) &= -(b_x(x\partial_t + t\partial_x) + b_y(y\partial_t + t\partial_y) + b_z(z\partial_t + t\partial_z))\end{aligned} \quad (54)$$

The quantities that multiply the r_i, b_j in Eq.(54) define the tensor L^μ_ν , namely,

$$L^j_k = -(x^j \partial_k - x^k \partial_j) \quad L^0_i = -(t \partial_i + x^i \partial_t) \quad (55)$$

Raising the lower index we obtain,

$$L^{\mu\nu} = x^\mu \partial^\nu - x^\nu \partial^\mu \quad (56)$$

(c.f. Maggiore, 2005, p.30, Eq.2.78). From Eq.(54) it is clear that, say $x \partial^y - y \partial^x$, is the z -component of the angular momentum. The angular momentum operator acts on a function $\psi(x)$ as follows $(\delta x^\mu = \Theta G_{rot}(\mathbf{r})^\mu_\nu x^\nu)$,

$$(1 + \Theta A_{rot}(\mathbf{r}, x))\psi(x) = \psi(x^\mu) - \delta x^\mu \partial_\mu \psi(x^\mu) = \psi(x^\mu - \delta x^\mu) = \psi'(x^\mu) \quad (57)$$

i.e. $\psi(x^\mu) \longrightarrow \psi'(x^\mu)$ and clearly $\psi(x^\mu) = \psi'(x'^\mu)$ where $x'^\mu = x^\mu + \delta x^\mu$. The base space for the angular momentum representation is the set of scalar functions.

5. Lorentz's Transformations of the Generators. It is of interest to study the Lorentz transformations properties of the generators $G_{rot}(\mathbf{r})$ and $G_{boost}(\mathbf{b})$, namely expressions as $\Lambda_{boost}(\mathbf{b})G_{rot}(\mathbf{e}_z)\Lambda_{boost}^{-1}(\mathbf{b})$ and $\Lambda_{rot}(\mathbf{r})G_{boost}(\mathbf{e}_z)\Lambda_{rot}^{-1}(\mathbf{r})$, c.f., Weinberg, 2005, p.60 (notice that here only rotations are associated with unitary operators). It is found,

$$\begin{aligned} \Lambda_{boost}(\mathbf{b})G_{rot}(\mathbf{e}_z)\Lambda_{boost}^{-1}(\mathbf{b}) &= G_{rot}(\mathbf{e}_z) - (\gamma - 1)(b_x b_z G_{rot}(\mathbf{e}_x) + b_y b_z G_{rot}(\mathbf{e}_y) + \\ & (b_z^2 - 1)G_{rot}(\mathbf{e}_z)) + v\gamma G_{boost}(\mathbf{b} \wedge \mathbf{e}_z) = \gamma G_{rot}(\mathbf{e}_z) - (\gamma - 1)G_{rot}(b_z \mathbf{b} \cdot \mathbf{e}) + v\gamma G_{boost}(\mathbf{b} \wedge \mathbf{e}_z) \end{aligned} \quad (58)$$

where v is the boost velocity and $\gamma = (1 - v^2)^{-1/2}$.

$$\begin{aligned} \Lambda_{rot}(\mathbf{r})G_{boost}(\mathbf{e}_z)\Lambda_{rot}^{-1}(\mathbf{r}) &= G_{boost}(\mathbf{e}_z) + G_{boost}(\mathbf{e}_x)(r_x r_z \Omega + r_y \sin \Theta) + \\ & + G_{boost}(\mathbf{e}_y)(r_y r_z \Omega - r_x \sin \Theta) + G_{boost}(\mathbf{e}_z)(r_z^2 - 1)\Omega = \\ & = (1 - \Omega)G_{boost}(\mathbf{e}_z) + G_{boost}(\Omega r_z \mathbf{r} \cdot \mathbf{e}) + G_{boost}(\sin \Theta \mathbf{r} \wedge \mathbf{r}_z) \end{aligned} \quad (59)$$

where $\Omega = 1 - \cos \Theta$. In the above equations, \mathbf{e} is a unit vector, and the notation should be transparent: $G_{boost}(\mathbf{b} \wedge \mathbf{e}_z) = b_y G_{boost}(\mathbf{e}_x) - b_x G_{boost}(\mathbf{e}_y)$ for example. Concerning Eq.(58) notice that it reduces to $\Lambda_{boost}(\mathbf{b})G_{rot}(\mathbf{e}_z)\Lambda_{boost}^{-1}(\mathbf{b}) = G_{rot}(\mathbf{e}_z)$ if $v = 0$, because γ is then equal to one, and also when $b_x = b_y = 0$ because then $b_z = 1$, i.e. the boosts are along the axis of rotation. In Equation (59) as well, $G_{boost}(\mathbf{e}_z)$ is left unchanged if Θ vanishes and also when the axis of the rotations is the z -axis.

5. Acknowledgements.

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6. References.

Maggiore, M.: *A Modern Introduction to Quantum Field Theory*, Oxford University Press, New York, (2005)

Weinberg, S.: *The Quantum Theory of Fields* Vol 1, Cambridge University Press, New York, (2005)